

UPPER AND LOWER ESTIMATES FOR SCHAUDER FRAMES AND ATOMIC DECOMPOSITIONS.

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ABSTRACT. We prove that a Schauder frame for any separable Banach space is shrinking if and only if it has an associated space with a shrinking basis, and that a Schauder frame for any separable Banach space is shrinking and boundedly complete if and only if it has a reflexive associated space. To obtain these results, we prove that the upper and lower estimate theorems for finite dimensional decompositions of Banach spaces can be extended and modified to Schauder frames. We show as well that if a separable infinite dimensional Banach space has a Schauder frame, then it also has a Schauder frame which is not shrinking.

1. INTRODUCTION

Frames for Hilbert spaces were introduced by Duffin and Schaeffer in 1952 [DS] to address some questions in non-harmonic Fourier series. However, the current popularity of frames is largely due to their successful application to signal processing, initiated by Daubechies, Grossmann, and Meyer in 1986 [DGM]. A *frame* for an infinite dimensional separable Hilbert space H is a sequence of vectors $(x_i)_{i=1}^{\infty} \subset H$ for which there exists constants $0 \leq A \leq B$ such that for any $x \in H$,

$$(1) \quad A\|x\|^2 \leq \sum_{i=1}^{\infty} |\langle x, x_i \rangle|^2 \leq B\|x\|^2.$$

If $A = B = 1$, then $(x_i)_{i=1}^{\infty}$ is called a *Parseval* frame. Given any frame $(x_i)_{i=1}^{\infty}$ for a Hilbert space H , there exists a frame $(f_i)_{i=1}^{\infty}$ for H , called an *alternate dual frame*, such that for all $x \in H$,

$$(2) \quad x = \sum_{i=1}^{\infty} \langle x, f_i \rangle x_i.$$

The equality in (2) allows the reconstruction of any vector x in the Hilbert space from the sequence of coefficients $(\langle x, f_i \rangle)_{i=1}^{\infty}$. The standard method to construct such a frame $(f_i)_{i=1}^{\infty}$ is to take $f_i = S^{-1}x_i$ for all $i \in \mathbb{N}$, where S is the positive, self-adjoint invertible operator on H

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defined by $Sx = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$ for all $x \in H$. The operator S is called the *frame operator* and the frame $(S^{-1}x_i)_{i=1}^{\infty}$ is called the *canonical dual frame* of $(x_i)_{i=1}^{\infty}$.

In their AMS memoir [HL], Han and Larson initiated studying the dilation viewpoint of frames. That is, analyzing frames as orthogonal projections of Riesz bases, where a Riesz basis is an unconditional basis for a Hilbert space. To start this approach, they proved the following theorem.

Theorem 1.1 ([HL]). *If $(x_i)_{i=1}^{\infty}$ is a frame for a Hilbert space H , then there exists a larger Hilbert space $Z \supset H$ and a Riesz basis $(z_i)_{i=1}^{\infty}$ for Z such that $P_X z_i = x_i$ for all $i \in \mathbb{N}$, where P_X is orthogonal projection onto X . Furthermore, if $(x_i)_{i=1}^{\infty}$ is Parseval, then $(z_i)_{i=1}^{\infty}$ can be taken to be an ortho-normal basis.*

The concept of a frame was extended to Banach spaces in 1989 by Grochenig [G] through the introduction of atomic decompositions. The main goal of the paper was to obtain for Banach spaces the unique association of a vector with the natural set of frame coefficients. In 2008, Schauder frames for Banach space were developed [CDOSZ] with the goal of creating a procedure to represent vectors using quantized coefficients. A Schauder frame essentially takes, as its definition, an extension of the equation (2) to Banach spaces.

Definition 1.2. Let X be an infinite dimensional separable Banach space. A sequence $(x_i, f_i)_{i=1}^{\infty} \subset X \times X^*$ is called a *Schauder frame* for X if $x = \sum_{i=1}^{\infty} f_i(x)x_i$ for all $x \in X$.

In particular, if $(x_i)_{i=1}^{\infty}$ and $(f_i)_{i=1}^{\infty}$ are frames for a Hilbert space H , then $(f_i)_{i=1}^{\infty}$ is an alternate dual frame for $(x_i)_{i=1}^{\infty}$ if and only if $(x_i, f_i)_{i=1}^{\infty}$ is a Schauder frame for H . As noted in [CDOSZ], a separable Banach space has a Schauder frame if and only if it has the bounded approximation property. By the uniform boundedness principle, for any Schauder frame $(x_i, f_i)_{i=1}^{\infty}$ of a Banach space X , there exists a constant $C \geq 1$ such that $\sup_{n \geq m} \|\sum_{i=m}^n f_i(x)x_i\| \leq C\|x\|$ for all $x \in X$. The least such value C is called the *frame constant* of $(x_i, f_i)_{i=1}^{\infty}$. A Schauder frame $(x_i, f_i)_{i=1}^{\infty}$ is called *unconditional* if the series $x = \sum_{i=1}^{\infty} f_i(x)x_i$ converges unconditionally for all $x \in X$. The following definition extends the notion of a basis being shrinking or boundedly complete to the context of frames.

Definition 1.3. Given a Schauder frame $(x_i, f_i)_{i=1}^{\infty} \subset X \times X^*$, let $T_n : X \rightarrow X$ be the operator $T_n(x) = \sum_{i \geq n} f_i(x)x_i$. The frame $(x_i, f_i)_{i=1}^{\infty}$ is called *shrinking* if $\|x^* \circ T_n\| \rightarrow 0$ for all $x^* \in X^*$. The frame $(x_i, f_i)_{i=1}^{\infty}$ is called *boundedly complete* if $\sum_{i=1}^{\infty} x^{**}(f_i)x_i$ converges in norm to an element of X for all $x^{**} \in X^{**}$.

As noted in [CL], if $(x_i)_{i=1}^{\infty}$ is a Schauder basis and $(x_i^*)_{i=1}^{\infty}$ are the biorthogonal functionals of $(x_i)_{i=1}^{\infty}$, then the frame $(x_i, x_i^*)_{i=1}^{\infty}$ is shrinking if and only if the basis $(x_i)_{i=1}^{\infty}$ is shrinking, and the frame $(x_i, x_i^*)_{i=1}^{\infty}$ is boundedly complete if and only if the basis $(x_i)_{i=1}^{\infty}$ is boundedly complete. Thus the definition of a frame being shrinking or boundedly complete is consistent with that of a basis. In [L], the frame properties shrinking and boundedly complete are called pre-shrinking and pre-boundedly complete. The following definitions allow the dilation viewpoint of Han and Larson to be extended to Schauder frames.

Definition 1.4. Let $(x_i, f_i)_{i=1}^\infty$ be a frame for a Banach space X and let Z be a Banach space with basis $(z_i)_{i=1}^\infty$ and coordinate functionals $(z_i^*)_{i=1}^\infty$. We call Z an *associated space* to $(x_i, f_i)_{i=1}^\infty$ and $(z_i)_{i=1}^\infty$ an *associated basis* if the operators $T : X \rightarrow Z$ and $S : Z \rightarrow X$ are bounded, where, $T(x) = T(\sum f_i(x)x_i) = \sum f_i(x)z_i$ for all $x \in X$ and $S(z) = S(\sum z_i^*(z)z_i) = \sum z_i^*(z)x_i$ for all $z \in Z$.

Essentially, Theorem 1.1 states that a frame for a Hilbert space has an associated basis which is a Riesz basis for a Hilbert space. Furthermore, the proof in [HL] actually involves constructing the operators T and S given in Definition 1.4. In [CDOSZ], it is shown that every Schauder frame has an associated space, which is referred to as the minimal associated space in [L]. Furthermore, the minimal associated basis will be unconditional if and only if the Schauder frame is unconditional. On the other hand, if (x_i, f_i) is a Schauder frame, then the reconstruction operator, S , for the minimal associated space contains c_0 in its kernel if and only if a finite number of vectors can be removed from (x_i) to make it a Schauder basis [LZ]. Thus, except in trivial cases, the minimal associated basis will not be boundedly complete. Given some desirable property for a basis to have, it is natural to consider the problem of characterizing whether or not a particular Schauder frame has an associated basis with that property. It is not difficult to see that if a Schauder frame has a shrinking associated basis, then the frame must be shrinking as well, and that if a Schauder frame has a boundedly complete associated basis, then the frame must be boundedly complete. Under additional assumptions, it is proven in [L] that the minimal associated space to a frame is shrinking if the frame itself is shrinking and that the maximal associated space to a frame is boundedly complete if the frame itself is boundedly complete. One of our main theorems is to give the following complete characterization.

Theorem 1.5. *Let $(x_i, f_i)_{i=1}^\infty$ be a Schauder frame for a Banach space X . Then $(x_i, f_i)_{i=1}^\infty$ is shrinking if and only if $(x_i, f_i)_{i=1}^\infty$ has a shrinking associated basis. Furthermore, $(x_i, f_i)_{i=1}^\infty$ is shrinking and boundedly complete if and only if $(x_i, f_i)_{i=1}^\infty$ has a reflexive associated space.*

To obtain Theorem 1.5, we prove a stronger result, involving quantitative bounds on the Szlenk index. The Szlenk index [Sz] is a co-analytic rank on the set of Banach spaces with separable dual, and was created to prove that there does not exist a single Banach space X with separable dual such that every Banach space with separable dual is isomorphic to a subspace of X . In particular, the Szlenk index of a Banach space is countable if and only if the Banach space has separable dual. In [OSZ2], it is shown that the higher order Tsirelson spaces $T_{\alpha,c}$, where α is a countable ordinal and $0 < c < 1$, can be used to measure the Szlenk index through the use of tree estimates. Furthermore, a Banach space with separable dual has Szlenk index at most $\omega^{\alpha\omega}$ for some given countable ordinal α , if and only if the Banach space satisfies subsequential $T_{\alpha,c}$ -upper tree estimates for some constant $0 < c < 1$. Thus proving theorems about upper tree estimates provides quantitative insight into Banach spaces with separable dual, and similarly, proving theorems about upper and lower tree estimates provides quantitative insight into separable reflexive Banach spaces. In [OSZ1], a characterization is given for when a separable reflexive Banach space embeds into a Banach space with an FDD

satisfying certain upper and lower block estimates, and in [FOSZ], a characterization is given for when a Banach space with separable dual embeds into a Banach space with an FDD satisfying certain upper block estimates. We extend both of those theorems to frames and by applying them to T_α and T_α^* , we obtain the following two characterizations.

Theorem 1.6. *Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X and let α be a countable ordinal. Then, the following are equivalent.*

- (a) *X has Szlenk index at most $\omega^{\alpha\omega}$.*
- (b) *X satisfies subsequential $T_{\alpha,c}$ -upper tree estimates for some constant $0 < c < 1$.*
- (c) *$(x_i, f_i)_{i=1}^\infty$ has an associated basis $(z_i)_{i=1}^\infty$ such that there exists $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and $0 < c < 1$ so that the FDD $(\text{span}_{j \in [n_i, n_{i+1})} z_j)_{i=1}^\infty$ satisfies subsequential $(t_{K_i})_{i=1}^\infty$ upper block estimates, where $(t_i)_{i=1}^\infty$ is the unit vector basis for $T_{\alpha,c}$.*

Theorem 1.7. *Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking and boundedly complete Schauder frame for a Banach space X . Then, the following are equivalent.*

- (a) *X and X^* both have Szlenk index at most $\omega^{\alpha\omega}$.*
- (b) *X satisfies subsequential $T_{\alpha,c}$ -upper tree estimates and subsequential $T_{\alpha,c}^*$ -lower tree estimates for some constant $0 < c < 1$.*
- (c) *$(x_i, f_i)_{i=1}^\infty$ has an associated basis $(z_i)_{i=1}^\infty$ such that there exists $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and $0 < c < 1$ so that the FDD $(\text{span}_{j \in [n_i, n_{i+1})} z_j)_{i=1}^\infty$ satisfies subsequential $(t_{K_i})_{i=1}^\infty$ upper block estimates and subsequential $(t_{K_i}^*)_{i=1}^\infty$ lower block estimates, where $(t_i)_{i=1}^\infty$ is the unit vector basis for $T_{\alpha,c}$.*

The Banach space $T_{\alpha,c}$ is reflexive, and thus the basis $(z_i)_{i=1}^\infty$ given in Theorem 1.6 is shrinking and the basis $(z_i)_{i=1}^\infty$ given in Theorem 1.7 is shrinking and boundedly complete. Thus Theorem 1.5 follows immediately from Theorems 1.6 and 1.7.

Both Schauder frames and atomic decompositions are natural extensions of frame theory into the study of Banach space. These two concepts are directly related, and some papers in the area such as [CHL], [CL], and [CLS] are stated in terms of atomic decompositions, while others such as [CDOSZ], [L], and [LZ] are stated in terms of Schauder frames.

Definition 1.8. Let X be a Banach space and Z be a Banach sequence space. We say that a sequence of pairs $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$ is an *atomic decomposition* of X with respect to Z if there exists positive constants A and B such that for all $x \in X$:

- (a) $(f_i(x_i))_{i=1}^\infty \in Z$,
- (b) $A\|x\| \leq \|(f_i(x_i))_{i=1}^\infty\|_Z \leq B\|x\|$,
- (c) $x = \sum_{i=1}^\infty f_i(x)x_i$.

If the unit vectors in the Banach space Z given in Definition 1.8 form a basis for Z , then an atomic decomposition is simply a Schauder frame with a specified associated space Z . We choose to use the terminology of Schauder frames for this paper instead of atomic decomposition as to us, an associated space is an object which is external to the space X and frame $(x_i, f_i)_{i=1}^\infty$. Our goals are essentially, to construct ‘nice’ associated spaces, given a particular Schauder

frame. However, our theorems can be stated in terms of atomic decompositions. In particular, Theorem 1.5 can be stated as the following.

Theorem 1.9. *Let X be a Banach space and Z be a Banach sequence space whose unit vectors form a basis for Z . Let $(x_i, f_i)_{i=1}^\infty$ be an atomic decomposition of X with respect to Z . Then $(x_i, f_i)_{i=1}^\infty$ is shrinking if and only if there exists a Banach sequence space Z' whose unit vectors form a shrinking basis for Z' such that $(x_i, f_i)_{i=1}^\infty$ is an atomic decomposition of X with respect to Z' . Furthermore, $(x_i, f_i)_{i=1}^\infty$ is shrinking and boundedly complete if and only if there exists a reflexive Banach sequence space Z' whose unit vectors form a basis for Z' such that $(x_i, f_i)_{i=1}^\infty$ is an atomic decomposition of X with respect to Z' .*

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2. SHRINKING AND BOUNDEDLY COMPLETE SCHAUDER FRAMES

It is well known that a basis (x_i) for a Banach space X is shrinking if and only if the biorthogonal functionals (x_i^*) form a boundedly complete basis for X^* . The following theorem extends this useful characterization to Schauder frames.

Theorem 2.1. [CL, Proposition 2.3][L, Proposition 4.8] *Let X be a Banach space with a Schauder frame $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$. The frame $(x_i, f_i)_{i=1}^\infty$ is shrinking if and only if $(f_i, x_i)_{i=1}^\infty$ is a boundedly complete Schauder frame for X^* .*

It is a classic and fundamental result of James that a basis for a Banach space is both shrinking and boundedly complete, if and only if the Banach space is reflexive. The following theorem shows that one side of James' characterization holds for frames.

Theorem 2.2. [CL, Proposition 2.4][L, Proposition 4.9] *If $(x_i, f_i)_{i=1}^\infty$ is a shrinking and boundedly complete Schauder frame of a Banach space X , then X is reflexive.*

It was left as an open question in [CL] whether the converse of Theorem 2.2 holds. The following theorem shows that this is false for any Banach space X , and is evidence of how general Schauder frames can exhibit fairly unintuitive structure.

Theorem 2.3. *Let X be a Banach space which admits a Schauder frame (i.e. has the bounded approximation property), then X has a Schauder frame which is not shrinking.*

Proof. Let $(x_i, f_i)_{i=1}^\infty$ be a Schauder frame for X . If $(x_i, f_i)_{i=1}^\infty$ is not shrinking, then we are done. Thus we assume that $(x_i, f_i)_{i=1}^\infty$ is shrinking. Fix $x \in X$ such that $x \neq 0$, and choose $(y_i^*)_{i=1}^\infty \subset S_{X^*}$ such that $y_i^* \rightarrow_{w^*} 0$. For all $n \in \mathbb{N}$, we define elements $(x'_{3n-2}, f'_{3n-2}), (x'_{3n-1}, f'_{3n-1}), (x'_{3n}, f'_{3n}) \in X \times X^*$, in the following way:

$$\begin{aligned} \bar{x}_{3n-2} &= x_n & \bar{x}_{3n-1} &= x & \bar{x}_{3n} &= x \\ \bar{f}_{3n-2} &= f_n & \bar{f}_{3n-1} &= -y_n^* & \bar{f}_{3n} &= y_n^* \end{aligned}$$

As $y_i^* \rightarrow_{w^*} 0$, it is not difficult to see that $(\bar{x}_i, \bar{f}_i)_{i=1}^\infty$ is a frame of X . However, $(\bar{x}_i, \bar{f}_i)_{i=1}^\infty$ is not shrinking. Indeed, let $x^* \in X^*$ such that $x^*(x) = 1$. As $(x_i, f_i)_{i=1}^\infty$ is shrinking, there exists

$N_0 \in \mathbb{N}$ such that $|\sum_{i=M}^{\infty} x^*(x_i) f_i(y)| \leq \frac{1}{4} \|y\|$ for all $y \in X$ and $M \geq N_0$. Let $M \geq N_0$ and choose $y \in B_X$ such that $y_M^*(y) > \frac{3}{4}$. We now have the following estimate,

$$\|x^* \circ T_{3M}\| \geq x^* \circ T_{3M}(y) = x^*(y) y_M^*(y) + \sum_{i=M+1}^{\infty} f_i(y) x^*(x_i) > \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

Thus we have that $\|x^* \circ T_N\| \not\rightarrow 0$, and hence $(\bar{x}_i, \bar{f}_i)_{i=1}^{\infty}$ is not shrinking. \square

As a Schauder frame must be shrinking in order to have a shrinking associated basis, Theorem 2.3 implies that not every Schauder frame for a reflexive Banach space has a reflexive associated space.

Definition 2.4. If $(x_i, f_i)_{i=1}^{\infty}$ is a Schauder frame for a Banach space X , and $(z_i)_{i=1}^{\infty}$ is an associated basis then $(x_i, f_i)_{i=1}^{\infty}$ is called strongly shrinking relative to $(z_i)_{i=1}^{\infty}$ if

$$\|x^* \circ S_n\| \rightarrow 0 \quad \text{for all } x^* \in X^*$$

where $S_n : Z \rightarrow X$ is defined by $S_n(z) = \sum_{i=n}^{\infty} z_i^*(z) x_i$.

It is clear that if a Schauder frame is strongly shrinking relative to some associated basis, then the Schauder frame must be shrinking. Also, if a Schauder frame has a shrinking associated basis, then the frame is strongly shrinking relative to the basis. In [CL], examples of shrinking Schauder frames are given which are not strongly shrinking relative to some given associated spaces. However, we will prove later that for any given shrinking Schauder frame, there exists an associated basis such that the frame is strongly shrinking relative to the basis. Before proving this, we state the following theorem which illustrates why the concept of strongly shrinking will be important to us and allows us to use frames in duality arguments.

Theorem 2.5. [CL, Lemma 1.7, Theorem 1.8] *If $(x_i, f_i)_{i=1}^{\infty}$ is a Schauder frame for a Banach space X , and $(z_i)_{i=1}^{\infty}$ is an associated basis for $(x_i, f_i)_{i=1}^{\infty}$ then $(z_i^*)_{i=1}^{\infty}$ is an associated basis for $(f_i, x_i)_{i=1}^{\infty}$, if and only if $(x_i, f_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i)_{i=1}^{\infty}$.*

Furthermore, given operators $T : X \rightarrow Z$ and $S : Z \rightarrow X$ defined by $T(x) = \sum f_i(x) z_i$ for all $x \in X$ and $S(z) = \sum z_i^(z) x_i$ for all $z \in Z$, if $(x_i, f_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i)_{i=1}^{\infty}$, then $S^* : X^* \rightarrow [z_i^*]$ and $T^* : [z_i^*] \rightarrow X^*$ are given by $S^*(x^*) = \sum x^*(x_i) z_i^*$ for all $x^* \in X^*$ and $T^*(z^*) = \sum z^*(z_i) f_i$ for all $z^* \in [z_i^*]$.*

Applying Theorem 2.5 to reflexive Banach spaces gives the following corollary.

Corollary 2.6. *If $(x_i, f_i)_{i=1}^{\infty}$ is a shrinking frame for a reflexive Banach space X and $(z_i)_{i=1}^{\infty}$ is an associated basis such that $(x_i, f_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i)_{i=1}^{\infty}$ then $(f_i, x_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i^*)_{i=1}^{\infty}$.*

Before proceeding further, we need some stability lemmas. Note that if $(z_i)_{i=1}^{\infty}$ is a basis for a Banach space Z , with projection operators $P_{(n,k)} : Z \rightarrow Z$ given by $P_{(n,k)}(\sum a_i z_i) = \sum_{i \in (n,k)} a_i z_i$, then $P_{(1,k)} \circ P_{(n,\infty)} = 0$ and $P_{(n,\infty)} \circ P_{(1,k)} = 0$ for all $k < n$. The analogous

property fails when working with frames. However, the following lemmas will essentially allow us to obtain this property within some given $\varepsilon > 0$ if n is chosen sufficiently larger than k .

Lemma 2.7. *Let $(x_i, f_i)_{i=1}^\infty$ be a Schauder frame for a Banach space X . Then for all $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $N > k$ and*

$$\sup_{n \geq m \geq N > k \geq n_0 \geq m_0} \left\| \sum_{i=m}^n f_i \left(\sum_{j=m_0}^{n_0} f_j(x) x_j \right) x_i \right\| \leq \varepsilon \|x\| \quad \text{for all } x \in X.$$

Proof. As $(x_i, f_i)_{i=1}^\infty$ is a Schauder frame, for each $1 \leq \ell \leq k$ with $f_\ell \neq 0$, there exists $N_\ell > k$ such that $\sup_{n \geq m \geq N_\ell} \left\| \sum_{i=m}^n f_i(x_\ell) x_i \right\| < \varepsilon / (k \|f_\ell\|)$. Let $N = \max_{1 \leq \ell \leq k} N_\ell$. We now obtain the following estimate for $n \geq m \geq N > k \geq n_0 \geq m_0$ and $x \in X$.

$$\begin{aligned} \left\| \sum_{i=m}^n \sum_{j=m_0}^{n_0} f_j(x) f_i(x_j) x_i \right\| &\leq k \max_{1 \leq \ell \leq k} \left\| \sum_{i=m}^n f_\ell(x) f_i(x_\ell) x_i \right\| \quad \text{as } k \geq n_0 \\ &\leq k \max_{1 \leq \ell \leq k} \left\| \sum_{i=m}^n f_i(x_\ell) x_i \right\| \|f_\ell\| \|x\| \leq \varepsilon \|x\| \quad \text{as } n \geq m \geq N_\ell. \end{aligned}$$

□

In terms of operators, Lemma 2.7 can be stated as for all $k \in \mathbb{N}$, if $(x_i, f_i)_{i=1}^\infty$ is a Schauder frame then $\lim_{N \rightarrow \infty} \|T_N \circ (Id_X - T_k)\| = 0$, where $T_n : X \rightarrow X$ is given by $T_n(x) = \sum_{i=1}^n f_i(x) x_i$ for all $n \in \mathbb{N}$. We now prove that for all $k \in \mathbb{N}$, if $(x_i, f_i)_{i=1}^\infty$ is a shrinking Schauder frame then $\lim_{N \rightarrow \infty} \|(Id_X - T_k) \circ T_N\| = 0$. The frame given in the proof of Theorem 2.3 shows that we cannot drop the condition of shrinking.

Lemma 2.8. *Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X . Then for all $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $N > k$ and*

$$\sup_{n \geq m \geq N > k \geq n_0 \geq m_0} \left\| \sum_{i=m_0}^{n_0} f_i \left(\sum_{j=m}^n f_j(x) x_j \right) x_i \right\| \leq \varepsilon \|x\| \quad \text{for all } x \in X.$$

Proof. By Theorem 2.1, $(f_i, x_i)_{i=1}^\infty$ is a Schauder frame for X^* . Thus for each $1 \leq \ell \leq k$ with $x_\ell \neq 0$, there exists $N_\ell > k$ such that $\sup_{n \geq m \geq N_\ell} \left\| \sum_{j=m}^n f_\ell(x_j) f_j \right\| < \varepsilon / (k \|x_\ell\|)$. Let $N = \max_{1 \leq \ell \leq k} N_\ell$. We now obtain the following estimate for $n \geq m \geq N > k \geq n_0 \geq m_0$ and

$x \in X$.

$$\begin{aligned}
\left\| \sum_{i=m_0}^{n_0} f_i \left(\sum_{j=m}^n f_j(x) x_j \right) x_i \right\| &\leq k \sup_{1 \leq \ell \leq k} \left\| f_\ell \left(\sum_{j=m}^n f_j(x) x_j \right) x_\ell \right\| \quad \text{as } k \geq n_0 \\
&= k \sup_{1 \leq \ell \leq k} \left| \sum_{j=m}^n f_j(x) f_\ell(x_j) \right| \|x_\ell\| \\
&\leq k \sup_{1 \leq \ell \leq k} \left\| \sum_{j=m}^n f_\ell(x_j) f_j \right\| \|x\| \|x_\ell\| \leq \varepsilon \|x\| \quad \text{as } n \geq m \geq N_\ell.
\end{aligned}$$

□

Our method for proving that every shrinking frame has a shrinking associated basis is to first prove that every shrinking frame is strongly shrinking with respect to some associated basis, and then renorm that associated basis to make it shrinking. The following theorem is thus our first major step.

Theorem 2.9. *Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X . Then $(x_i, f_i)_{i=1}^\infty$ has an associated basis $(z_i)_{i=1}^\infty$ such that $(x_i, f_i)_{i=1}^\infty$ is strongly shrinking relative to $(z_i)_{i=1}^\infty$.*

Proof. We repeatedly apply Lemma 2.8 to obtain a subsequence $(N_k)_{k=1}^\infty$ of \mathbb{N} such that for all $k \in \mathbb{N}$,

$$(3) \quad \sup_{n \geq m \geq N_k} \left\| \sum_{i=1}^k f_i \left(\sum_{j=m}^n f_j(x) x_j \right) x_i \right\| \leq 2^{-2k} \|x\| \quad \text{for all } x \in X.$$

We assume without loss of generality that $x_i \neq 0$ for all $i \in \mathbb{N}$. We denote the unit vector basis of c_{00} by $(z_i)_{i=1}^\infty$, and define the following norm, $\|\cdot\|_Z$ for all $(a_i) \in c_{00}$.

$$(4) \quad \left\| \sum a_i z_i \right\|_Z = \max_{n \geq m} \left\| \sum_{i=m}^n a_i x_i \right\| \vee \max_{k \in \mathbb{N}; n \geq m \geq N_k} 2^k \left\| \sum_{i=1}^k f_i \left(\sum_{j=m}^n a_j x_j \right) x_i \right\|.$$

It follows easily that $(z_i)_{i=1}^\infty$ is a bimonotone basic sequence, and thus $(z_i)_{i=1}^\infty$ is a bimonotone basis for the completion of c_{00} under $\|\cdot\|_Z$, which we denote by Z . We first prove that $(z_i)_{i=1}^\infty$ is an associated basis for $(x_i, f_i)_{i=1}^\infty$. Let C be the frame constant of $(x_i, f_i)_{i=1}^\infty$. That is, $\max_{n \geq m} \left\| \sum_{i=m}^n f_i(x) x_i \right\| \leq C \|x\|$ for all $x \in X$. By (3) and (4), the operator $T : X \rightarrow Z$, defined by $T(x) = \sum f_i(x) z_i$ for all $x \in X$, is bounded and $\|T\| \leq C$. We have that $\left\| \sum_{i=m}^n a_i z_i \right\|_Z \geq \left\| \sum_{i=m}^n a_i x_i \right\|$, and hence the operator $S : Z \rightarrow X$ defined by $S(z) = \sum z_i^*(z) x_i$ is bounded and $\|S\| = 1$. Thus we have that $(z_i)_{i=1}^\infty$ is an associated basis for $(x_i, f_i)_{i=1}^\infty$.

We now prove that $(x_i, f_i)_{i=1}^\infty$ is strongly shrinking relative to $(z_i)_{i=1}^\infty$. Let $\varepsilon > 0$ and $x^* \in B_{X^*}$. As $(x_i, f_i)_{i=1}^\infty$ is shrinking, we may choose $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon/2$ and $\left\| \sum_{j=k+1}^\infty x^*(x_j) f_j \right\| <$

$\varepsilon/2$. We obtain the following estimate for any $N \geq N_k$ and $z = \sum a_i z_i \in Z$.

$$\begin{aligned}
x^*\left(\sum_{i=N}^{\infty} a_i x_i\right) &= \sum_{j=1}^{\infty} x^*(x_j) f_j \left(\sum_{i=N}^{\infty} a_i x_i\right) \quad \text{as } (f_i, x_i)_{i=1}^{\infty} \text{ is a frame for } X^* \\
&= \sum_{j=1}^k x^*(x_j) f_j \left(\sum_{i=N}^{\infty} a_i x_i\right) + \sum_{j=k+1}^{\infty} x^*(x_j) f_j \left(\sum_{i=N}^{\infty} a_i x_i\right) \\
&\leq \|x^*\| \left\| \sum_{j=1}^k f_j \left(\sum_{i=N}^{\infty} a_i x_i\right) x_j \right\| + \left\| \sum_{j=k+1}^{\infty} x^*(x_j) f_j \right\| \|z\|_Z \quad \text{as } \left\| \sum_{i=N}^{\infty} a_i x_i \right\| \leq \|z\|_Z \\
&\leq \|x^*\| 2^{-k} \|z\|_Z + \left\| \sum_{j=k+1}^{\infty} x^*(x_j) f_j \right\| \|z\|_Z \quad \text{by (4) as } N \geq N_k \\
&< \varepsilon/2 \|z\|_Z + \varepsilon/2 \|z\|_Z
\end{aligned}$$

We thus have that for all $x^* \in X^*$ and $\varepsilon > 0$, that there exists $M \in \mathbb{N}$ such that $|x^*(\sum_{i=N}^{\infty} z_i^*(z) x_i)| < \varepsilon$ for all $N \geq M$ and $z^* \in B_{Z^*}$. Hence, $(x_i, f_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i)_{i=1}^{\infty}$. \square

The following lemmas incorporate an associated basis into the tail and initial segment estimates of Lemma 2.7 and Lemma 2.8.

Lemma 2.10. *Let X be a Banach space with a shrinking Schauder frame $(x_i, f_i)_{i=1}^{\infty}$. Let Z be a Banach space with basis $(z_i)_{i=1}^{\infty}$ such that $(x_i, f_i)_{i=1}^{\infty}$ is strongly shrinking relative to $(z_i)_{i=1}^{\infty}$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\sup_{k \geq n \geq m} \left\| \sum_{i=N}^{\infty} \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| < \varepsilon \|x^*\| \quad \text{for all } x^* \in X^*$$

Proof. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. By renorming Z , we may assume without loss of generality that $(z_i)_{i=1}^{\infty}$ is bimonotone. Let $K \geq 1$ be the frame constant of the frame $(f_i, x_i)_{i=1}^{\infty}$ for X^* . We choose a finite $\frac{\varepsilon}{2\|S\|}$ -net $(y_{\alpha}^*)_{\alpha \in A}$ in $\{y^* \in K \cdot B_{Y^*} : y^* \in \text{span}_{1 \leq i \leq k}(f_i)\}$. By Theorem 2.5, the bounded operator $S^* : X^* \rightarrow [z_i^*]$ is given by $S^*(x^*) = \sum_{i=1}^{\infty} x^*(x_i) z_i^*$ for all $x^* \in X^*$. As $(z_i^*)_{i=1}^{\infty}$ is a basis for $[z_i^*]_{i=1}^{\infty}$, for each $\alpha \in A$, there exists $N_{\alpha} \in \mathbb{N}$ such that $\|P_{[N_{\alpha}, \infty)} \circ S^*(y_{\alpha}^*)\| = \|\sum_{i=N_{\alpha}}^{\infty} y_{\alpha}^*(x_i) z_i^*\| < \frac{\varepsilon}{2}$. We set $N = \max_{\alpha \in A} N_{\alpha}$. Given, $x^* \in B_X^*$ and $m, n \in \mathbb{N}$ such that $k \geq n \geq m$, we choose $\alpha \in A$ such that $\|y_{\alpha}^* - \sum_{j=m}^n x^*(x_j) f_j\| < \frac{\varepsilon}{2\|S\|}$, which yields the following

estimates.

$$\begin{aligned}
\left\| \sum_{i=N}^{\infty} \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| &= \|P_{[N,\infty)} \circ S^* \left(\sum_{j=m}^n x^*(x_j) f_j \right)\| \\
&\leq \|P_{[N,\infty)} \circ S^*(y_\alpha^*)\| + \|P_{[N,\infty)} \circ S^*(y_\alpha^* - \sum_{j=m}^n x^*(x_j) f_j)\| \\
&\leq \|P_{[N,\infty)} \circ S^*(y_\alpha^*)\| + \|P_{[N,\infty)}\| \|S\| \|y_\alpha^* - \sum_{j=m}^n x^*(x_j) f_j\| \\
&< \frac{\varepsilon}{2} + \|S\| \frac{\varepsilon}{2\|S\|} = \varepsilon
\end{aligned}$$

□

Lemma 2.11. *Let X be a Banach space with a shrinking Schauder frame $(x_i, f_i)_{i=1}^\infty$ and let Z be a Banach space with a basis $(z_i)_{i=1}^\infty$. Then for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$\sup_{n \geq m \geq N} \left\| \sum_{i=1}^k \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| < \varepsilon \|x^*\| \quad \text{for all } x^* \in X^*$$

Proof. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. As $(x_j, f_j)_{j=1}^\infty$ is a Schauder frame for X , the series $\sum_{j=1}^\infty f_j(x_i) x_j$ converges in norm to x_i for all $i \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that $\sup_{n \geq m \geq N} \left\| \sum_{j=m}^n f_j(x_i) x_j \right\| < \frac{\varepsilon}{k\|z_i^*\|}$ for all $1 \leq i \leq k$. For $x^* \in B_{X^*}$ and $n \geq m \geq N$, we have that

$$\left\| \sum_{i=1}^k \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| \leq \sum_{i=1}^k \left\| \sum_{j=m}^n f_j(x_i) x_j \right\| \|z_i^*\| < \sum_{i=1}^k \frac{\varepsilon}{k\|z_i^*\|} \|z_i^*\| = \varepsilon$$

□

The following lemma and theorem are based on an idea of W. B. Johnson [J], and are analogous to Proposition 3.1 in [FOSZ], and Lemma 4.3 in [OS]. Their importance comes from allowing us to use arguments that require ‘skipping coordinates’, and in particular, will allow us to apply Proposition 2.14.

Lemma 2.12. *Let X be a Banach space with a boundedly complete Schauder frame $(x_i, f_i)_{i=1}^\infty \subset X \times X^*$. Let $\varepsilon_i \searrow 0$ and $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$. There exists $(k_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that for all $x^{**} \in X^{**}$ and for all $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $k_N < M < k_{N+1}$ and*

$$\sup_{p_{M+1} > n \geq m \geq p_{M-1}} \left\| \sum_{i=m}^n x^{**}(f_i) x_i \right\| < \varepsilon_N \|x^{**}\|.$$

Proof. Assume not, then there exists $\varepsilon > 0$ and $K_0 \in \mathbb{N}$ such that for all $K > K_0$ there exists $x_K^{**} \in B_{X^{**}}$ such that for all $K_0 < M < K$ there exists $n_{K,M}, m_{K,M} \in \mathbb{N}$ with $p_{M-1} \leq m_{K,M} \leq n_{K,M} < p_{M+1}$ and $\|\sum_{i=m_{K,M}}^{n_{K,M}} x_K^{**}(f_i)x_i\| > \varepsilon$. As $[p_{M-1}, p_{M+1}]$ is finite, we may choose a sequence $(K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that for every $M \in \mathbb{N}$ there exists $n_M, m_M \in \mathbb{N}$ such that $n_{K_i,M} = n_M$ and $m_{K_i,M} = m_M$ for all $i \geq M$. After passing to a further subsequence of $(K_i)_{i=1}^\infty$, we may assume that there exists $x^{**} \in X^{**}$ such that $x_{K_i}^{**}(f_j) \rightarrow x^{**}(f_j)$ for all $j \in \mathbb{N}$. Thus $\|\sum_{i=m_M}^{n_M} x^{**}(f_i)x_i\| \geq \varepsilon$. This contradicts that the series $\sum_{i=1}^\infty x^{**}(f_i)x_i$ is norm convergent. \square

Theorem 2.13. *Let X be a Banach space with a shrinking Schauder frame $(x_i, f_i)_{i=1}^\infty$. Let Z be a Banach space with basis $(z_i)_{i=1}^\infty$ such that $(x_i, f_i)_{i=1}^\infty$ is strongly shrinking relative to $(z_i)_{i=1}^\infty$. Let $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and $(\delta_i)_{i=1}^\infty \subset (0, 1)$ with $\delta_i \searrow 0$. Then there exists $(q_i)_{i=1}^\infty, (N_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that for any $(k_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ and $y^* \in S_{X^*}$, there exists $y_i^* \in X^*$ and $t_i \in (N_{k_{i-1}-1}, N_{k_{i-1}})$ for all $i \in \mathbb{N}$ with $N_0 = 0$ and $t_0 = 0$ so that the following hold*

- (a) $y^* = \sum_{i=1}^\infty y_i^*$
and for all $\ell \in \mathbb{N}$ we have
- (b) either $\|y_\ell^*\| < \delta_\ell$ or $\sup_{p_{q_{t_\ell-1}} \geq n \geq m} \|\sum_{j=m}^n y_\ell^*(x_j)f_j\| < \delta_\ell \|y_\ell^*\|$ and
 $\sup_{n \geq m \geq p_{q_{t_\ell}}} \|\sum_{j=m}^n y_\ell^*(x_j)f_j\| < \delta_\ell \|y_\ell^*\|,$
- (c) $\|P_{[p_{q_{N_{k_\ell}}, p_{q_{N_{k_\ell+1}}}]}) \circ S^*(y_{\ell-1}^* + y_\ell^* + y_{\ell+1}^* - y^*)\|_{Z^*} < \delta_\ell,$

where P_I is the projection operator $P_I : [z_i^*] \rightarrow [z_i^*]$ given by $P_I(\sum a_i z_i^*) = \sum_{i \in I} a_i z_i^*$ for all $\sum a_i z_i^* \in [z_i^*]$ and all intervals $I \subseteq \mathbb{N}$.

Proof. By Theorems 2.1 and 2.5, $(f_i, x_i)_{i=1}^\infty$ is a boundedly complete frame for X^* with associated basis $(z_i^*)_{i=1}^\infty$. After renorming, we may assume without loss of generality that $(z_i^*)_{i=1}^\infty$ is bimonotone. We let K be the frame constant of $(f_i, x_i)_{i=1}^\infty$. Let $\varepsilon_i \searrow 0$ such that $2\varepsilon_{i+1} < \varepsilon_i < \delta_i$ and $(1 + K)\varepsilon_i < \delta_{i+1}^2$ for all $i \in \mathbb{N}$.

By repeatedly applying Lemma 2.8 to the frame $(x_i, f_i)_{i=1}^\infty$ of X , we may choose $(q_k)_{k=1}^\infty \in [\mathbb{N}]^\omega$ such that for all $k \in \mathbb{N}$,

$$(5) \quad \sup_{n \geq m \geq p_{q_{k+1}} > p_{q_k} \geq n_0 \geq m_0} \left\| \sum_{i=m_0}^{n_0} f_i \left(\sum_{j=m}^n f_j(x)x_j \right) x_i \right\| \leq \varepsilon_k \|x\| \quad \text{for all } x \in X.$$

By Lemma 2.11, after possibly passing to a subsequence of $(q_k)_{k=1}^\infty$, we may assume that for all $k \in \mathbb{N}$,

$$(6) \quad \sup_{n \geq m \geq p_{q_{k+1}}} \left\| \sum_{i=1}^{p_{q_k}} \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| \leq \varepsilon_k \|x^*\| \quad \text{for all } x^* \in X^*.$$

By applying Lemma 2.7 to the frame $(x_i, f_i)_{i=1}^\infty$ of X , after possibly passing to a subsequence of $(q_k)_{k=1}^\infty$, we may assume that for all $k \in \mathbb{N}$,

$$(7) \quad \sup_{n \geq m \geq p_{q_{k+1}} > p_{q_k} \geq n_0 \geq m_0} \left\| \sum_{i=m}^n f_i \left(\sum_{j=m_0}^{n_0} f_j(x) x_j \right) x_i \right\| \leq \varepsilon_{k+1} \|x\| \quad \text{for all } x \in X.$$

By Lemma 2.10, after possibly passing to a subsequence of $(q_k)_{k=1}^\infty$, we may assume that for all $k \in \mathbb{N}$,

$$(8) \quad \sup_{p_{q_k} > n \geq m \geq 1} \left\| \sum_{i=p_{q_{k+1}}}^\infty \left(\sum_{j=m}^n x^*(x_j) f_j(x_i) \right) z_i^* \right\| \leq \varepsilon_{k+1} \|x^*\| \quad \text{for all } x^* \in X^*.$$

By Lemma 2.12, there exists $(N_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ such that $N_0 = 0$ and for all $x^* \in X^*$ and for all $k \in \mathbb{N}$ there exists $t_k \in \mathbb{N}$ such that $N_k < t_k < N_{k+1}$ and $\sup_{p_{q_{t_k-1}} \leq n \leq m < p_{q_{t_k+1}}} \left\| \sum_{i=n}^m x^*(x_i) f_i \right\| < \varepsilon_k \|x^*\|$.

Let $(k_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ and $y^* \in S_{X^*}$. For each $i \in \mathbb{N}$, we choose $t_i \in (N_{k_i}, N_{k_{i+1}})$ with $t_0 = 1$ such that

$$(9) \quad \sup_{p_{q_{t_i+1}} > n \geq m \geq p_{q_{t_i-1}}} \left\| \sum_{j=m}^n y^*(x_j) f_j \right\| < \varepsilon_i.$$

We now set $y_i^* = \sum_{j=p_{q_{t_i-1}}}^{p_{q_{t_i}}-1} y^*(x_j) f_j$ for all $i \in \mathbb{N}$. We have that,

$$\sum_{i=1}^\infty y_i^* = \sum_{i=1}^\infty \sum_{j=p_{q_{t_i-1}}}^{p_{q_{t_i}}-1} y^*(x_j) f_j = \sum_{j=1}^\infty y^*(x_j) f_j = y^*.$$

Thus (a) is satisfied. In order to prove (b), we let $\ell \in \mathbb{N}$ and assume that $\|y_\ell^*\| > \delta_\ell$. Let $m, n \in \mathbb{N}$ such that $n \geq m \geq p_{q_{t_\ell}}$. To prove property (b), we consider the following inequalities.

$$\begin{aligned} \left\| \sum_{j=m}^n y_\ell^*(x_j) f_j \right\| &= \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_\ell-1}}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i(x_j) f_j \right\| \\ &\leq \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_\ell-1}}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i(x_j) f_j \right\| + \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_\ell-1}}}^{p_{q_{t_\ell-1}}-1} y^*(x_i) f_i(x_j) f_j \right\| \\ &\leq K \left\| \sum_{i=p_{q_{t_\ell-1}}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i \right\| + \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_\ell-1}}}^{p_{q_{t_\ell-1}}-1} y^*(x_i) f_i(x_j) f_j \right\| \\ &< K \varepsilon_{t_\ell} + \varepsilon_{t_\ell} \quad \text{by (9) and (7)} \\ &< (1 + K) \varepsilon_{t_\ell} \|y_\ell^*\| / \delta_\ell < (1 + K) \varepsilon_\ell \|y_\ell^*\| / \delta_\ell < \delta_\ell \|y_\ell^*\|. \end{aligned}$$

Thus $\sup_{n \geq m \geq p_{q_{t_\ell}}} \|\sum_{j=m}^n y_\ell^*(x_j) f_j\| < \delta_\ell \|y_\ell^*\|$, proving one of the inequalities in (b). We now assume that $\ell > 1$, and let $m, n \in \mathbb{N}$ such that $p_{q_{t_{\ell-1}}} \geq n \geq m$. To prove the remaining inequality in (b), we consider the following.

$$\begin{aligned}
\left\| \sum_{j=m}^n y_\ell^*(x_j) f_j \right\| &= \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_{\ell-1}}}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i(x_j) f_j \right\| \\
&\leq \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_{\ell-1}}+1}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i(x_j) f_j \right\| + \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_{\ell-1}}}}^{p_{q_{t_{\ell-1}}+1}-1} y^*(x_i) f_i(x_j) f_j \right\| \\
&\leq \left\| \sum_{j=m}^n \sum_{i=p_{q_{t_{\ell-1}}+1}}^{p_{q_{t_\ell}}-1} y^*(x_i) f_i(x_j) f_j \right\| + K \left\| \sum_{i=p_{q_{t_{\ell-1}}}}^{p_{q_{t_{\ell-1}}+1}-1} y^*(x_i) f_i \right\| \\
&< \varepsilon_{t_{\ell-1}+1} + K \varepsilon_{\ell-1} \quad \text{by (5) and (9)} \\
&< (\varepsilon_{t_{\ell-1}+1} + K \varepsilon_{\ell-1}) \|y_\ell^*\| / \delta_\ell < (1 + K) \varepsilon_{\ell-1} \|y_\ell^*\| / \delta_\ell < \delta_\ell \|y_\ell^*\|.
\end{aligned}$$

Thus $\sup_{p_{q_{t_{\ell-1}}} \geq n \geq m} \|\sum_{j=m}^n y_\ell^*(x_j) f_j\| < \delta_\ell \|y_\ell^*\|$, and hence all of (b) is satisfied. To prove (c), we now consider the following,

$$\begin{aligned}
\|P_{[p_{q_{N_{k_\ell}}}, p_{q_{N_{k_{\ell+1}}})} S^*(y_{\ell-1}^* + y_\ell^* + y_{\ell+1}^* - y^*)\|_{Z^*} &= \left\| \sum_{i=p_{q_{N_{k_\ell}}}}^{p_{q_{N_{k_{\ell+1}}}}-1} (y_{\ell-1}^* + y_\ell^* + y_{\ell+1}^* - y^*)(x_i) z_i^* \right\| \\
&= \left\| \sum_{i=p_{q_{N_{k_\ell}}}}^{p_{q_{N_{k_{\ell+1}}}}-1} \left(\sum_{j=1}^{p_{q_{t_{\ell-2}}}-1} y^*(x_j) f_j(x_i) + \sum_{j=p_{q_{t_{\ell+1}}}}^{\infty} y^*(x_j) f_j(x_i) \right) z_i^* \right\| \\
&\leq \left\| \sum_{i=p_{q_{N_{k_\ell}}}}^{p_{q_{N_{k_{\ell+1}}}}-1} \left(\sum_{j=1}^{p_{q_{t_{\ell-2}}}-1} y^*(x_j) f_j(x_i) \right) z_i^* \right\| + \left\| \sum_{i=p_{q_{N_{k_\ell}}}}^{p_{q_{N_{k_{\ell+1}}}}-1} \left(\sum_{j=p_{q_{t_{\ell+1}}}}^{\infty} y^*(x_j) f_j(x_i) \right) z_i^* \right\| \\
&\leq \left\| \sum_{i=p_{q_{N_{k_\ell}}}}^{\infty} \left(\sum_{j=1}^{p_{q_{t_{\ell-2}}}-1} y^*(x_j) f_j(x_i) \right) z_i^* \right\| + \left\| \sum_{i=1}^{p_{q_{N_{k_{\ell+1}}}}} \left(\sum_{j=p_{q_{t_{\ell+1}}}}^{\infty} y^*(x_j) f_j(x_i) \right) z_i^* \right\| \\
&< \varepsilon_{N_{k_\ell}-1} + \varepsilon_{N_{k_{\ell+1}}} < \varepsilon_\ell \quad \text{by (8) and (6).}
\end{aligned}$$

Thus (c) is satisfied. \square

Properties of coordinate systems for Banach spaces such as frames, bases and FDDs can impose certain structure on infinite dimensional subspaces. For our purposes, this structure

can be intrinsically characterized in terms of even trees of vectors [OSZ1]. In order to index even trees, we define

$$T_{\infty}^{\text{even}} = \{(n_1, \dots, n_{2\ell}) : n_1 < \dots < n_{2\ell} \text{ are in } \mathbb{N} \text{ and } \ell \in \mathbb{N}\}.$$

If X is a Banach space, an indexed family $(x_{\alpha})_{\alpha \in T_{\infty}^{\text{even}}} \subset X$ is called an *even tree*. Sequences of the form $(x_{(n_1, \dots, n_{2\ell-1}, k)})_{k=n_{2\ell-1}+1}^{\infty}$ are called *nodes*. This should not be confused with the more standard terminology where a node would refer to an individual member of the tree. Sequences of the form $(n_{2\ell-1}, x_{(n_1, \dots, n_{2\ell})})_{\ell=1}^{\infty}$ are called *branches*. A *normalized tree*, i.e. one with $\|x_{\alpha}\| = 1$ for all $\alpha \in T_{\infty}^{\text{even}}$, is called *weakly null* (or w^* -null) if every node is a weakly null (or w^* -null) sequence.

Given $1 > \varepsilon > 0$ and $A \subset (\mathbb{N} \times S_{X^*})^{\omega}$, we let $A_{\varepsilon} = \{(l_i, y_i^*) \in (\mathbb{N} \times S_{X^*})^{\omega} : \exists (k_i, x_i^*) \in A \text{ such that } k_i \leq l_i, \|x_i^* - y_i^*\| < \varepsilon 2^{-i} \forall i \in \mathbb{N}\}$, and we let $\overline{A}_{\varepsilon}$ be the closure of A_{ε} in $(\mathbb{N} \times S_{X^*})^{\omega}$. We consider the following game between players S (subspace chooser) and P (point chooser). The game has an infinite sequence of moves; on the n^{th} move S picks $k_n \in \mathbb{N}$ and a cofinite dimensional w^* -closed subspace Z_n of X^* and P responds by picking an element $x_n^* \in S_{X^*}$ such that $d(x_n^*, Z_n) < \varepsilon 2^{-n}$. S wins the game if the sequence $(k_i, x_i^*)_{i=1}^{\infty}$ the players generate is an element of $\overline{A}_{5\varepsilon}$, otherwise P is declared the winner. This is referred to as the (A, ε) -game and was introduced in [OSZ1]. The following proposition is essentially an extension of Proposition 2.6 in [FOSZ] from FDDs to frames, and relates properties of w^* -null even trees and winning strategies of the (A, ε) -game to blockings of a frame.

Proposition 2.14. *Let X be an infinite-dimensional Banach space with a shrinking Schauder frame $(x_i, f_i)_{i=1}^{\infty}$. Let $A \subseteq (\mathbb{N} \times S_{X^*})^{\omega}$. The following are equivalent.*

- (1) *For all $\varepsilon > 0$ there exists $(K_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ and $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \searrow 0$ and $(p_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ such that if $(y_i^*)_{i=1}^{\infty} \subset S_{X^*}$ and $(r_i)_{i=0}^{\infty} \in [\mathbb{N}]^{\omega}$ so that $\sup_{p_{r_{i-1}+1} \geq n \geq m \geq 1} \|\sum_{j=m}^n y_i^*(x_j) f_j\| < \delta_i$ and $\sup_{n \geq m \geq p_{r_i}} \|\sum_{j=m}^n y_i^*(x_j) f_j\| < \delta_i$ for all $i \in \mathbb{N}$ then $(K_{r_{i-1}}, y_i^*) \in \overline{A}_{\varepsilon}$.*
- (2) *For all $\varepsilon > 0$, S has a winning strategy for the (A, ε) -game.*
- (3) *For all $\varepsilon > 0$ every normalized w^* -null even tree in X^* has a branch in $\overline{A}_{\varepsilon}$.*

Proof. The equivalences (2) \iff (3) are given in [FOSZ].

We now assume (1) holds, and will prove (3). Let $\varepsilon > 0$ and let $(x_{(n_1, \dots, n_{2\ell})}^*)_{(n_1, \dots, n_{2\ell}) \in T_{\infty}^{\text{even}}}$ be a w^* -null even tree in X^* . There exists $(K_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ and $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \searrow 0$ and $(p_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ such that if $(y_i^*)_{i=1}^{\infty} \subset S_{X^*}$ and $(r_i)_{i=0}^{\infty} \in [\mathbb{N}]^{\omega}$ so that $\sup_{p_{r_{i-1}+1} \geq n \geq m \geq 1} \|\sum_{j=m}^n y_i^*(x_j) f_j\| < \delta_i$ and $\sup_{n \geq m \geq p_{r_i}} \|\sum_{j=m}^n y_i^*(x_j) f_j\| < \delta_i$ for all $i \in \mathbb{N}$ then $(K_{r_{i-1}}, y_i^*) \in \overline{A}_{\varepsilon}$.

We shall construct by induction sequences $(r_i)_{i=0}^{\infty}, (n_i)_{i=1}^{\infty} \in [\mathbb{N}]^{\omega}$ such that $K_{r_i} = n_{2i+1}$ and $\sup_{p_{r_{i-1}+1} \geq n \geq m \geq 1} \|\sum_{j=m}^n x_{(n_1, \dots, n_{2i})}^*(x_j) f_j\| < \delta_i$ and $\sup_{n \geq m \geq p_{r_i}} \|\sum_{j=m}^n y_i^*(x_j) f_j\| < \delta_i$ for all $i \in \mathbb{N}$. To start, we let $r_0 = 1$ and $n_1 = K_1$. Now, if $\ell \in \mathbb{N}$ and $(r_i)_{i=0}^{\ell}$ and $(n_i)_{i=1}^{2\ell+1}$ have been chosen, then using that $(x_{(n_1, \dots, n_{2\ell+1}, k)})_{k=n_{2\ell+1}+1}^{\infty}$ is w^* -null, we may choose $n_{2\ell+2} > n_{2\ell+1}$ such that $\|x_{(n_1, \dots, n_{2\ell+1}, n_{2\ell+2})}^*(x_j) f_j\| < (p_{r_{\ell}} + 1)^{-1} \delta_{\ell+1}$. Thus, $\sup_{p_{r_{\ell}+1} \geq n \geq m \geq 1} \|\sum_{j=m}^n x_{(n_1, \dots, n_{2\ell+1}, n_{2\ell+2})}^*(x_j) f_j\| <$

$\delta_{\ell+1}$. As $(x_j, f_j)_{j=1}^\infty$ is a Schauder frame, we may choose $r_{\ell+1} > r_\ell$ such that

$$\sup_{n \geq m \geq p_{r_{\ell+1}}} \left\| \sum_{j=m}^n x_{(n_1, \dots, n_{2\ell+1}, n_{2\ell+2})}^* (x_j) f_j \right\| < \delta_{\ell+1}.$$

We then let $n_{2\ell+2} = K_{r_{\ell+1}}$. Thus, our sequences $(r_i)_{i=0}^\infty$ and $(n_i)_{i=1}^\infty$ may be constructed by induction to satisfy the desired properties, giving us that $(n_{2i-1}, x_{(n_1, \dots, n_{2i})}^*)_{i=1}^\infty = (K_{r_{i-1}}, x_{(n_1, \dots, n_{2i})}^*)_{i=1}^\infty \in \overline{A}_\varepsilon$.

We now assume (2) holds, and will prove (1). Let $\varepsilon > 0$ and assume that player S has a winning strategy for the (A, ε) -game. That is, there exists an indexed collection $(k_{(x_1^*, \dots, x_\ell^*)})_{(x_1^*, \dots, x_\ell^*) \in X^{<\mathbb{N}}}$ of natural numbers, and an indexed collection $(X_{(x_1^*, \dots, x_\ell^*)}^*)_{(x_1^*, \dots, x_\ell^*) \in X^{<\mathbb{N}}}$ of co-finite dimensional w^* -closed subsets of X^* such that if $(x_i^*)_{i=1}^\infty \subset S_{X^*}$ and $d(x_i^*, X_{(x_1^*, \dots, x_i^*)}^*) < \frac{1}{10}\varepsilon 2^{-i}$ for all $i \in \mathbb{N}$ then $(k_{(x_1^*, \dots, x_i^*)}, X_{(x_1^*, \dots, x_i^*)}^*)_{i=1}^\infty \in \overline{A}_{\varepsilon/2}$ and $(k_{(x_1^*, \dots, x_i^*)})_{i=1}^\infty \in [\mathbb{N}]^\omega$.

We construct by induction $(K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, $(\delta_i)_{i=1}^\infty \in (0, 1)^\omega$ and a nested collection $(D_i)_{i=1}^\infty \subset [X^{<\omega}]^\omega$ such that D_i is $\frac{1}{20}\varepsilon 2^{-i}$ -dense in $[f_j]_{j=1}^{p_i}$ and if $(y_i^*)_{i=1}^\infty \subset S_{X^*}$ and $(r_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ so that $\sup_{p_{r_{i-1}+1} \geq n \geq m \geq 1} \left\| \sum_{j=m}^n y_i^*(x_j) f_j \right\| < \delta_i$ and $\sup_{n \geq m \geq p_{r_i}} \left\| \sum_{j=m}^n y_i^*(x_j) f_j \right\| < \delta_i$ for all $i \in \mathbb{N}$, and $x_i^* \in D_{r_{i-1}}$ such that $\|y_i^* - x_i^*\| < \frac{1}{20}\varepsilon 2^{-i}$ for all $i \in \mathbb{N}$, then $K_{r_{i-1}} \geq k_{(x_1^*, \dots, x_{i-1}^*)}$, and $d(x_i^*, X_{(x_1^*, \dots, x_i^*)}^*) < \frac{1}{10}\varepsilon 2^{-i}$. This would give that $(k_{(x_1^*, \dots, x_i^*)}, X_{(x_1^*, \dots, x_i^*)}^*)_{i=1}^\infty \in \overline{A}_{\varepsilon/2}$. Hence, $(K_{r_{i-1}}, y_i^*)_{i=1}^\infty \in \overline{A}_\varepsilon$ as $K_{r_{i-1}} \geq k_{(x_1^*, \dots, x_{i-1}^*)}$ and $\|y_i^* - x_i^*\| < \frac{1}{20}\varepsilon 2^{-i}$ for all $i \in \mathbb{N}$. Thus all that remains is to show that $(K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, and $(D_i)_{i=1}^\infty \subset [X^{<\omega}]^\omega$ may be constructed inductively with the desired properties.

We start by choosing $K_1 = k_\emptyset$. As $(x_i, f_i)_{i=1}^\infty$ is a shrinking Schauder frame for X and $X_\emptyset^* \subset X^*$ is co-finite dimensional and w^* -closed, by Lemma 2.8 there exists $p_1 \in \mathbb{N}$ and $\delta_1 > 0$ such that if $\sup_{p_1 \geq n \geq m \geq 1} \left\| \sum_{j=m}^n y^*(x_j) f_j \right\| < \delta_1$ for some $y^* \in S_{X^*}$ then $d(y^*, X_\emptyset^*) < \frac{1}{20}\varepsilon$. We then let D_1 be some finite $\frac{1}{20}\varepsilon$ -net in $[f_i]_{i=1}^{p_1}$. Now we assume $n \in \mathbb{N}$ and that $(K_i)_{i=1}^n \in [\mathbb{N}]^{<\omega}$, $(p_i)_{i=1}^n \in [\mathbb{N}]^{<\omega}$, $(\delta_i)_{i=1}^n \in (0, 1)^{<\omega}$ and $(D_i)_{i=1}^n \subset [X^{<\omega}]^{<\omega}$ have been suitably chosen. As $(x_i, f_i)_{i=1}^\infty$ is a shrinking Schauder frame for X and $X_{(x_1^*, \dots, x_\ell^*)}^* \subset X^*$ is co-finite dimensional and w^* -closed for all $(x_1^*, \dots, x_\ell^*) \in [D_n]^{<\omega}$, by Lemma 2.8 there exists $p_{n+1} \in \mathbb{N}$ and $\delta_{n+1} > 0$ such that if $\sup_{p_{n+1} \geq n \geq m \geq 1} \left\| \sum_{j=m}^n y^*(x_j) f_j \right\| < \delta_{n+1}$ for some $y^* \in S_{X^*}$ then $d(y^*, \cap_{(x_1^*, \dots, x_\ell^*) \in [D_n]^{<\omega}} X_{(x_1^*, \dots, x_\ell^*)}^*) < \frac{1}{20}\varepsilon 2^{-n-1}$. We then let $K_{n+1} = \max_{(x_1^*, \dots, x_\ell^*) \in [D_n]^{<\omega}} k_{(x_1^*, \dots, x_\ell^*)}$ and let D_{n+1} be a finite $\frac{1}{20}\varepsilon 2^{-n-1}$ -net in $[f_i]_{i=1}^{p_{n+1}}$. \square

3. UPPER AND LOWER ESTIMATES

Let X be a Banach space, $V = (v_i)_{i=1}^\infty$ be a normalized 1-unconditional basis, and $1 \leq C < \infty$. We say that X satisfies *subsequential C - V -upper tree estimates* if every weakly null even tree $(x_\alpha)_{\alpha \in T_{\text{even}}^\infty}$ in X has a branch $(n_{2\ell-1}, x_{(n_1, \dots, n_{2\ell})})_{\ell=1}^\infty$ such that $(x_{(n_1, \dots, n_{2\ell})})_{\ell=1}^\infty$ is C -dominated by $(v_{n_{2\ell-1}})_{\ell=1}^\infty$. We say that X satisfies *subsequential V -upper tree estimates* if it satisfies subsequential C - V -upper tree estimates for some $1 \leq C < \infty$. If X is a subspace of a dual space, we say that X satisfies *subsequential C - V -lower w^* tree estimates* if every

w^* -null even tree $(x_\alpha)_{\alpha \in T_\infty^{\text{even}}}$ in X has a branch $(n_{2\ell-1}, x_{(n_1, \dots, n_{2\ell})})_{\ell=1}^\infty$ such that $(x_{(n_1, \dots, n_{2\ell})})_{\ell=1}^\infty$ C -dominates $(v_{n_{2\ell-1}})_{\ell=1}^\infty$.

A basic sequence $V = (v_i)_{i=1}^\infty$ is called *C-right dominant* if for all sequences $m_1 < m_2 < \dots$ and $n_1 < n_2 < \dots$ of positive integers with $m_i \leq n_i$ for all $i \in \mathbb{N}$ the sequence $(v_{m_i})_{i=1}^\infty$ is C -dominated by $(v_{n_i})_{i=1}^\infty$. We say that $(v_i)_{i=1}^\infty$ is *right dominant* if for some $C \geq 1$ it is C -right dominant.

Lemma 3.1. [FOSZ, Lemma 2.7] *Let X be a Banach space with separable dual, and let $V = (v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, right dominant basic sequence. If X satisfies subsequential V -upper tree estimates, then X^* satisfies subsequential V^* -lower w^* tree estimates.*

Let Z be a Banach space with an FDD $(E_i)_{i=1}^\infty$, let $V = (v_i)_{i=1}^\infty$ be a normalized 1-unconditional basis, and let $1 \leq C < \infty$. We say that $(E_i)_{i=1}^\infty$ satisfies *subsequential C - V -upper block estimates* if every normalized block sequence $(z_i)_{i=1}^\infty$ of $(E_i)_{i=1}^\infty$ in Z is C -dominated by $(v_{m_i})_{i=1}^\infty$, where $m_i = \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that $(E_i)_{i=1}^\infty$ satisfies *subsequential C - V -lower block estimates* if every normalized block sequence $(z_i)_{i=1}^\infty$ of $(E_i)_{i=1}^\infty$ in Z C -dominates $(v_{m_i})_{i=1}^\infty$, where $m_i = \min \text{supp}_E(z_i)$ for all $i \in \mathbb{N}$. We say that $(E_i)_{i=1}^\infty$ satisfies *subsequential V -upper (or lower) block estimates* if it satisfies subsequential C - V -upper (or lower) block estimates for some $1 \leq C < \infty$.

Note that if $(E_i)_{i=1}^\infty$ satisfies subsequential C - V -upper block estimates and $(z_i)_{i=1}^\infty$ is a normalized block sequence with $\max \text{supp}_E(z_{i-1}) < k_i \leq \min \text{supp}_E(z_i)$ for all $i > 1$, then $(z_i)_{i=1}^\infty$ is C -dominated by $(v_{k_i})_{i=1}^\infty$ (and a similar remark holds for lower estimates).

Subsequential $V^{(*)}$ -upper block estimates and subsequential V -lower block estimates are dual properties, as shown in the following proposition from [OSZ1].

Proposition 3.2. [OSZ1, Proposition 2.14] *Assume that Z has an FDD $(E_i)_{i=1}^\infty$, and let $V = (v_i)_{i=1}^\infty$ be a normalized and 1-unconditional basic sequence. The following statements are equivalent:*

- (a) $(E_i)_{i=1}^\infty$ satisfies subsequential V -lower block estimates in Z .
- (b) $(E_i^*)_{i=1}^\infty$ satisfies subsequential $V^{(*)}$ -upper block estimates in $Z^{(*)}$.

(Here subsequential $V^{(*)}$ -upper estimates are with respect to $(v_i^*)_{i=1}^\infty$, the sequence of biorthogonal functionals to $(v_i)_{i=1}^\infty$).

Moreover, if $(E_i)_{i=1}^\infty$ is bimonotone in Z , then the equivalence holds true if one replaces, for some $C \geq 1$, V -lower estimates by C - V -lower estimates in (a) and $V^{(*)}$ -upper estimates by C - $V^{(*)}$ -upper estimates in (b).

Note that by duality, Proposition 3.2 holds if we interchange the words “upper” and “lower”.

Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X , and let (v_i) be a normalized, 1-unconditional, block stable, 1-right dominant, and shrinking basic sequence. For any $C > 0$, we may apply Proposition 2.14 to the set $A = \{(n_i, x_i^*)_{i=1}^\infty \in (\mathbb{N} \times S_{X^*})^\omega : (x_i^*)_{i=1}^\infty \text{ } C\text{-dominates } (v_{n_i}^*)_{i=1}^\infty\}$ to obtain the following corollary.

Corollary 3.3. *Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X , and let $V = (v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, block stable, right dominant, and shrinking basic sequence. The following are equivalent.*

- (1) *There exists $C > 0$, $(K_i)_{i=1}^\infty, (p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, and $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \searrow 0$ such that if $(y_i^*)_{i=1}^\infty \subset S_{X^*}$ and $(r_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ so that $\sup_{1 \leq n \leq m \leq p_{r_{i-1}+1}} \|\sum_{j=n}^m y_i^*(x_j) f_j\| < \delta_i$ and $\sup_{p_{r_i} \leq n \leq m} \|\sum_{j=n}^m y_i^*(x_j) f_j\| < \delta_i$ then $(y_i^*) \succeq_C (v_{K_{r_{i-1}}}^*)$.*
- (2) *X satisfies subsequential V upper tree estimates.*

Let Z be a Banach space with a basis $(z_i)_{i=1}^\infty$, let $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, and let $V = (v_i)_{i=1}^\infty$ be a normalized 1-unconditional basic sequence. The space $Z_V(p_i)$ is defined to be the completion of c_{00} with respect to the following norm $\|\cdot\|_{Z_V}$:

$$\left\| \sum a_i z_i \right\|_{Z_V} = \max_{M \in \mathbb{N}, 1 \leq r_0 \leq r_1 < \dots < r_M} \left\| \sum_{i=1}^M \left\| \sum_{j=p_{r_i}}^{p_{r_{i+1}}-1} a_j z_j \right\|_Z v_{r_i} \right\|_V \quad \text{for } (a_i) \in c_{00}.$$

The following proposition from [OSZ1] is what makes the space Z_V essential for us. Recall that in [OSZ1] a basic sequence, $(v_i)_{i=1}^\infty$, is called C -block stable for some $C \geq 1$ if any two normalized block bases $(x_i)_{i=1}^\infty$ and $(y_i)_{i=1}^\infty$ with $\max(\text{supp}(x_i), \text{supp}(y_i)) < \min(\text{supp}(x_{i+1}), \text{supp}(y_{i+1}))$ for all $i \in \mathbb{N}$ are C -equivalent. We say that $(v_i)_{i=1}^\infty$ is *block stable* if it is C -block stable for some constant C . We will make use of the fact that the property of block stability dualizes. That is, if $(v_i)_{i=1}^\infty$ is a block stable basic sequence then $(v_i^*)_{i=1}^\infty$ is also a block stable basic sequence. Another simple, though important, consequence of a normalized basic sequence $(v_i)_{i=1}^\infty$ being block stable, is that there exists a constant $c \geq 1$ such that $(v_{n_i})_{i=1}^\infty$ is c -equivalent to $(v_{n_{i+1}})_{i=1}^\infty$ for all $(n_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$. Block stability has been considered before in various forms and under different names. In particular, it has been called the blocking principle [CJT] and the shift property [CK] (see [FR] for alternative forms). The following proposition recalls some properties of $Z_V(p_i)$ which were shown in [OSZ1].

Proposition 3.4. [OSZ1, Corollary 3.2, Lemmas 3.3, 3.5, and 3.6] *Let $V = (v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, and C -block stable basic sequence. If Z is a Banach space with a basis $(z_i)_{i=1}^\infty$ and $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, then $(z_i)_{i=1}^\infty$ satisfies $2C$ - V -lower block estimates in $Z_V(p_i)$. If the basis $(v_i)_{i=1}^\infty$ is boundedly complete then $(z_i)_{i=1}^\infty$ is a boundedly complete basis for $Z_V(p_i)$. If the basis $(v_i)_{i=1}^\infty$ is shrinking and $(z_i)_{i=1}^\infty$ is shrinking in Z , then $(z_i)_{i=1}^\infty$ is a shrinking basis for $Z_V(p_i)$.*

If $U = (u_i)_{i=1}^\infty$ is a normalized, 1-unconditional and block-stable basic sequence such that $(v_i)_{i=1}^\infty$ is dominated by $(u_i)_{i=1}^\infty$ and $(z_i)_{i=1}^\infty$ satisfies subsequential U -upper block estimates in Z , then $(z_i)_{i=1}^\infty$ also satisfies subsequential U -upper block estimates in $Z_V(p_i)$.

Theorem 3.5. *Let X be a Banach space with a shrinking Schauder frame (x_i, f_i) which is strongly shrinking relative to some Banach space Z with basis (z_i) and bounded operators $T : X \rightarrow Z$ and $S : Z \rightarrow X$ defined by $T(x) = \sum f_i(x) z_i$ for all $x \in X$ and $S(z) = \sum z_i^*(z) x_i$ for*

all $z \in Z$. Let $V = (v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, block stable, right dominant, and shrinking basic sequence. If X satisfies subsequential V upper tree estimates, then there exists $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that $Z_{(v_{K_i}^*)}^*(n_i)$ is an associated space of $(f_i, x_i)_{i=1}^\infty$ with bounded operators $S^* : X^* \rightarrow Z_{(v_{K_i}^*)}^*(n_i)$ and $T^* : Z_{(v_{K_i}^*)}^*(n_i) \rightarrow X$ given by $S^*(x^*) = \sum x^*(x_i)z_i^*$ for all $x^* \in X^*$ and $S^*(z^*) = \sum z^*(z_i)f_i$ for all $z^* \in Z_{(v_{K_i}^*)}^*(n_i)$.

Proof. After renorming, we may assume that the basis $(z_i)_{i=1}^\infty$ is bimonotone. The sequence $(f_i, x_i)_{i=1}^\infty$ is a boundedly complete Schauder frame for X^* by Theorem 2.1, and we have that X^* satisfies subsequential V^* lower w^* tree estimates by Lemma 3.1. By Theorem 2.5, the basis $(z_i^*)_{i=1}^\infty$ is an associated basis for $(f_i, x_i)_{i=1}^\infty$ with bounded operators $S^* : X^* \rightarrow Z^*$ and $T^* : Z^* \rightarrow X$ given by $S^*(x^*) = \sum x^*(x_i)z_i^*$ for all $x^* \in X^*$ and $S^*(z^*) = \sum z^*(z_i)f_i$ for all $z^* \in Z^*$. Let $\varepsilon > 0$. By Corollary 3.3, there exists $C > 0$, $(K_i)_{i=1}^\infty, (p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, and $\bar{\delta} = (\delta_i) \subset (0, 1)$ with $\delta_i \searrow 0$ and $\sum \delta_i < \varepsilon$ such that if $(y_i^*)_{i=1}^\infty \subset S_{X^*}$ and $(r_i)_{i=0}^\infty \in [\mathbb{N}]^\omega$ so that $\sup_{1 \leq n \leq m \leq p_{r_{i-1}+1}} \|\sum_{j=n}^m y_i^*(x_j)f_j\| < \delta_i$ and $\sup_{p_{r_i} \leq n \leq m} \|\sum_{j=n}^m y_i^*(x_j)f_j\| < \delta_i$ then $(y_i^*) \succeq_C (v_{K_{r_{i-1}}}^*)$. We apply Theorem 2.13 to $(x_i, f_i)_{i=1}^\infty$, $(p_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, and $(\delta_i)_{i=1}^\infty \subset (0, 1)$ to obtain $(q_i), (N_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ satisfying the conclusion of Theorem 2.13.

By Theorem 2.5, $(z_i^*)_{i=1}^\infty$ is an associated basis for $(f_i, x_i)_{i=1}^\infty$, and we denote the norm on Z^* by $\|\cdot\|_{Z^*}$. We block the basis $(z_i^*)_{i=1}^\infty$ into an FDD by setting $E_i = \text{span}_{j \in [p_{q_{N_i}}, p_{q_{N_{i+1}}})} z_j^*$ for all $i \in \mathbb{N}$. We now define a new norm $\|\cdot\|_{\bar{Z}^*}$ on $\text{span}(z_i^*)_{i=1}^\infty$ by

$$\left\| \sum a_i z_i^* \right\|_{\bar{Z}^*} = \max_{M \in \mathbb{N}, 1 \leq r_0 \leq r_1 < \dots < r_M} \left\| \sum_{i=1}^M \left\| \sum_{j=p_{q_{N_{r_i}}}}^{p_{q_{N_{r_{i+1}}}}-1} a_j z_j^* \right\|_{Z^*} v_{K_{N_{r_i}}}^* \right\|_{V^*} \quad \text{for all } (a_i) \in c_{00}.$$

We let \bar{Z}^* be the completion of $\text{span}(z_i^*)_{i=1}^\infty$ under the norm $\|\cdot\|_{\bar{Z}^*}$. Note that $\|z^*\|_{Z^*} \leq \|z^*\|_{\bar{Z}^*}$ for all $z^* \in Z^*$. As $(v_i^*)_{i=1}^\infty$ is block stable, there exists a constant $c \geq 1$ such that $(v_{n_i}^*)_{i=1}^\infty \approx_c (v_{n_{i+1}}^*)_{i=1}^\infty$ for all $(n_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$. We now show that $\|S^*(y^*)\|_{\bar{Z}^*} \leq (1+2\varepsilon)3cC\|y^*\|$ for all $y^* \in X^*$.

Let $y^* \in X^*$, $M \in \mathbb{N}$, and $1 \leq r_0 \leq r_1 < \dots < r_M$. We will show that $(1+2\varepsilon)3cC\|S^*(y^*)\| \geq \left\| \sum_{i=1}^M \left\| \sum_{j=p_{q_{N_{r_i}}}}^{p_{q_{N_{r_{i+1}}}}-1} y^*(x_j)z_j^* \right\|_{Z^*} v_{K_{N_{r_i}}}^* \right\|_{V^*}$. By Theorem 2.13, there exists $y_i^* \in X^*$ and $t_i \in (N_{r_{i-1}-1}, N_{r_i-1})$ for all $i \in \mathbb{N}$ with $N_0 = 0$ and $t_0 = 0$ such that

- (a) $y^* = \sum_{i=1}^\infty y_i^*$
and for all $i \in \mathbb{N}$ we have
- (b) either $\|y_i^*\| < \delta_i$ or $\sup_{p_{q_{t_{i-1}}} \geq n \geq m} \|\sum_{j=m}^n y_i^*(x_j)f_j\| < \delta_i\|y_i^*\|$ and
 $\|\sup_{n \geq m \geq p_{q_{t_i}}} \|\sum_{j=m}^n y_i^*(x_j)f_j\| < \delta_i\|y_i^*\|$,
- (c) $\|P_{[p_{q_{N_{k_i}}}, p_{q_{N_{k_{i+1}}})]}^* \circ S^*(y_{i-1}^* + y_i^* + y_{i+1}^* - y^*)\|_{Z^*} < \delta_i$

We let $A = \{i \in \mathbb{N} : \|y_i^*\| > \delta_i\}$. By our choice of $(p_i)_{i=1}^\infty$, we have that $(y_i^*/\|y_i^*\|)_{i \in A} \succeq_C (v_{K_{r_{i-1}}}^*)_{i \in A}$. Thus, $C\|\sum_{i \in A} y_i^*\| \geq \|\sum_{i \in A} \|y_i^*\| v_{K_{r_{i-1}}}^*\|_{V^*}$. We now obtain the following lower

estimate for $\|y^*\|$.

$$\begin{aligned}
\|y^*\| &= \left\| \sum_{i \in A} y_i^* \right\| && \text{by (a)} \\
&\geq \left\| \sum_{i \in A} y_i^* \right\| + \sum_{i \notin A} \|y_i^*\| - \varepsilon && \text{as } \sum \delta_i < \varepsilon \\
&\geq \frac{1}{C} \left\| \sum_{i \in A} \|y_i^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} + \sum_{i \notin A} \|y_i^*\| - \varepsilon && \text{as } C \left\| \sum_{i \in A} y_i^* \right\| \geq \left\| \sum_{i \in A} \|y_i^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} \\
&\geq \frac{1}{C} \left\| \sum_{i \in A} \|y_i^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} - \varepsilon \\
&\geq \frac{1}{3cC} (\left\| \sum_{i \in A} \|y_i^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} + \left\| \sum_{i \in A} \|y_{i-1}^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} + \left\| \sum_{i \in A} \|y_{i+1}^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*}) - \varepsilon \\
&\geq \frac{1}{3cC} \left\| \sum_{i \in A} \|y_{i-1}^* + y_i^* + y_{i+1}^*\| v_{K_{r_{i-1}}}^* \right\|_{V^*} - \varepsilon \\
&\geq \frac{1}{3cC \|S\|} \left\| \sum_{i \in A} \|S^*(y_{i-1}^* + y_i^* + y_{i+1}^*)\|_{Z^*} v_{K_{r_{i-1}}}^* \right\|_{V^*} - \varepsilon && \text{as } (v_i^*) \text{ is 1-unconditional} \\
&\geq \frac{1}{3cC \|S\|} \left\| \sum_{i \in A} \|P_{[p_{q_{N_i}}, p_{q_{N_{i+1}}})}^* S^*(y_{i-1}^* + y_i^* + y_{i+1}^*)\|_{Z^*} v_{K_{r_{i-1}}}^* \right\|_{V^*} - \varepsilon && \text{as } (z_i) \text{ is bimonotone} \\
&\geq \frac{1}{3cC \|S\|} \left\| \sum_{j=p_{q_{N_i}}}^{p_{q_{N_{i+1}}}-1} y^*(x_j) z_j \right\| v_{K_{r_{i-1}}}^* \right\|_{V^*} - 2\varepsilon && \text{by (c)}
\end{aligned}$$

Thus we have that $\|S^* y^*\|_{\bar{Z}^*} \leq (1 + \varepsilon) 3cC \|S\| \|y^*\|$ for all $y^* \in X^*$. Hence, $S^* : X^* \rightarrow \bar{Z}^*$ is an isomorphism. We have that $T^* : Z^* \rightarrow X^*$ is bounded, and hence $T^* : \bar{Z}^* \rightarrow X^*$ is bounded as well, as $\|z^*\|_{Z^*} \leq \|z^*\|_{\bar{Z}^*}$ for all $z^* \in Z^*$. Thus, \bar{Z}^* is an associated space of X^* . \square

The following theorem can be thought of as an extension of Theorem 1.1 in [FOSZ] to frames.

Theorem 3.6. *Let X be a Banach space with a shrinking Schauder frame $(x_i, f_i)_{i=1}^\infty$. Let $(v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, block stable, right dominant, and shrinking basic sequence. If X satisfies subsequential $(v_i)_{i=1}^\infty$ upper tree estimates, then there exists $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and an associated space Z with a shrinking basis $(z_i)_{i=1}^\infty$ such that the FDD $(\text{span}_{j \in [n_i, n_{i+1})} z_i)_{i=1}^\infty$ satisfies subsequential $(v_{K_i})_{i=1}^\infty$ upper block estimates.*

Proof. As $(x_i, f_i)_{i=1}^\infty$ is a shrinking Schauder frame, it is strongly shrinking relative to some associated basis $(z_i)_{i=1}^\infty$ for a Banach space Z by Theorem 2.9. We thus have bounded operators $T : X \rightarrow Z$ and $S : Z \rightarrow X$ defined by $T(x) = \sum f_i(x) z_i$ for all $x \in X$ and $S(z) = \sum z_i^*(z) x_i$ for all $z \in Z$. By Theorem 3.5, $Z_{(v_{K_i}^*)}^*(n_i)$ is an associated space of $(f_i, x_i)_{i=1}^\infty$ with bounded operators $S^* : X^* \rightarrow Z_{(v_{K_i}^*)}^*(n_i)$ and $T^* : Z_{(v_{K_i}^*)}^*(n_i) \rightarrow X$ given by $S^*(x^*) = \sum x^*(x_i) z_i^*$ for all $x^* \in X^*$ and $S^*(z^*) = \sum z^*(z_i) f_i$ for all $z^* \in Z_{(v_{K_i}^*)}^*(n_i)$. We define \bar{Z} as the completion

of $[z_i]_{i=1}^\infty$ under the norm $\|\sum a_i z_i\|_{\bar{Z}} = \sup_{z^* \in B_{Z_{(v_{K_i}^*)}(n_i)}} z^*(\sum a_i z_i)$. As $(z_i^*)_{i=1}^\infty$ is a boundedly complete basis of $Z_{(v_{K_i}^*)}(n_i)$ by Lemma 3.4, we have that $(z_i)_{i=1}^\infty$ is a shrinking basis for \bar{Z} and that the dual of \bar{Z} is $Z_{(v_{K_i}^*)}(n_i)$. We now prove that \bar{Z} is an associated space of $(x_i, f_i)_{i=1}^\infty$.

If $(x_i^*)_{i=1}^\infty \subset X^*$ and $x_i^* \rightarrow_{w^*} 0$ then $(S^*(x_i^*))_{i=1}^\infty$ converges w^* to 0 as a sequence in Z^* . Thus $((S^*(x_i^*))(z_j))_{i=1}^\infty$ converges to 0 for all $j \in \mathbb{N}$. Lemma 3.4 gives that $(z_i^*)_{i=1}^\infty$ is a boundedly complete basis for $Z_{(v_{K_i}^*)}(n_i)$, and hence converging w^* to 0 in $Z_{(v_{K_i}^*)}(n_i)$ is equivalent to converging coordinate wise to 0. Hence, $(S^*(x_i^*))_{i=1}^\infty$ converges w^* to 0 in $Z_{(v_{K_i}^*)}(n_i)$. Thus $S^* : X^* \rightarrow Z_{(v_{K_i}^*)}(n_i)$ is w^* to w^* continuous, and hence is a dual operator. Thus, $S : \bar{Z} \rightarrow X$.

If $(z_i^*)_{i=1}^\infty \subset Z_{(v_{K_i}^*)}(n_i)$ converges w^* to 0 in $Z_{(v_{K_i}^*)}(n_i)$, then $(z_i^*)_{i=1}^\infty$ converges coordinate wise to 0, and hence $(z_i^*)_{i=1}^\infty$ converges w^* to 0 in Z^* . Thus, $(T^*(z_i^*))_{i=1}^\infty \rightarrow_{w^*} 0$ in X^* . We thus have that $T^* : Z_{(v_{K_i}^*)}(n_i) \rightarrow X^*$ is w^* to w^* continuous, and is hence a dual operator. Thus, $T : X \rightarrow \bar{Z}$ is bounded. This gives us that, \bar{Z} is an associated space for $(x_i, f_i)_{i=1}^\infty$. By Lemma 3.4 we have that the FDD $(\text{span}_{j \in [p_{N_i}, p_{N_{i+1}}]} z_j^*)_{i=1}^\infty$ satisfies $(v_{K_i}^*)$ lower block estimates in \bar{Z}^* , and hence $(\text{span}_{j \in [p_{N_i}, p_{N_{i+1}}]} z_j)_{i=1}^\infty$ satisfies $(v_{K_i})_{i=1}^\infty$ upper block estimates in \bar{Z} . \square

The following theorem can be thought of as an extension of Theorem 4.6 in [OSZ1] to frames.

Theorem 3.7. *Let X be a Banach space with a shrinking and boundedly complete Schauder frame $(x_i, f_i)_{i=1}^\infty$. Let $(u_i)_{i=1}^\infty$ be a normalized, 1-unconditional, block stable, right dominant, and shrinking basic sequence, and let $(v_i)_{i=1}^\infty$ be a normalized, 1-unconditional, block stable, left dominant, and shrinking basic sequence such that (u_i) dominates (v_i) . Then X satisfies subsequential $(u_i, v_i)_{i=1}^\infty$ tree estimates, if and only if there exists $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and a reflexive associated space Z with a basis $(z_i)_{i=1}^\infty$ such that the FDD $(\text{span}_{j \in [n_i, n_{i+1}]} z_j)_{i=1}^\infty$ satisfies subsequential $(u_{K_i}, v_{K_i})_{i=1}^\infty$ block estimates.*

Proof. By Theorem 3.6, $(x_i, f_i)_{i=1}^\infty$ has an associated basis $(z_i)_{i=1}^\infty$ such that there exists $(m_i)_{i=1}^\infty, (k_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ and an associated space Z with a shrinking basis $(z_i)_{i=1}^\infty$ such that the FDD $(\text{span}_{j \in [n_i, n_{i+1}]} z_j)_{i=1}^\infty$ satisfies subsequential $(u_{K_i})_{i=1}^\infty$ upper block estimates. We have that $(f_i, x_i)_{i=1}^\infty$ is a shrinking frame for X^* which is strongly shrinking relative to the associated basis $(z_i^*)_{i=1}^\infty$ by Corollary 2.6. The space X satisfying subsequential $(v_i)_{i=1}^\infty$ lower tree estimates implies that X^* satisfies subsequential $(v_i^*)_{i=1}^\infty$ upper tree estimates. Thus we may apply Theorem 3.5 to the space X^* , the frame $(f_i, x_i)_{i=1}^\infty$ and the associated basis $(z_i^*)_{i=1}^\infty$ to obtain $(n_i)_{i=1}^\infty, (K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$ such that $Z_{(v_{K_i}^*)}(n_i)$ is an associated space of $(x_i, f_i)_{i=1}^\infty$. Furthermore, we may assume that $(n_i)_{i=1}^\infty$ is a subsequence of $(m_i)_{i=1}^\infty$ and that $(K_i)_{i=1}^\infty$ is a subsequence of $(k_i)_{i=1}^\infty$ as $(v_i)_{i=1}^\infty$ is left dominant. By Lemma 3.4, the FDD $(\text{span}_{j \in [n_i, n_{i+1}]} z_j)_{i=1}^\infty$ satisfies subsequential $(u_{K_i}, v_{K_i})_{i=1}^\infty$ block estimates. \square

We now show that Theorem 1.6 follows immediately from Theorem 3.6. For the same reason, Theorem 1.7 follows immediately from Theorem 3.7.

Proof of Theorem 1.6. Let $(x_i, f_i)_{i=1}^\infty$ be a shrinking Schauder frame for a Banach space X and let α be a countable ordinal. The equivalences (a) \iff (b) are given in [OSZ2].

Let $(t_i)_{i=1}^\infty$ be the unit vector basis for $T_{\alpha,c}$, where $0 < c < 1$ is some constant. If a Banach space has an FDD satisfying subsequential $(t_{K_i})_{i=1}^\infty$ upper block estimates for some sequence $(K_i)_{i=1}^\infty \in [\mathbb{N}]^\omega$, then it satisfies subsequential $T_{\alpha,c}$ -upper tree estimates. Thus if $(x_i, f_i)_{i=1}^\infty$ has an associated space with an FDD satisfying subsequential $(t_{K_i})_{i=1}^\infty$ upper block estimates, then X embeds into a Banach space satisfying subsequential $T_{\alpha,c}$ -upper tree estimates. Hence, X itself would satisfy subsequential $T_{\alpha,c}$ -upper tree estimates. Thus (c) \implies (b).

The unit vector basis, $(t_i)_{i=1}^\infty$, for $T_{\alpha,c}$ is a normalized, 1-unconditional, block stable, right dominant, and shrinking basic sequence. Thus (b) \implies (c) by Theorem 3.6. \square

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